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Geometry and analysis on isospectral sets. I. Riemannian geometry, asymptotic case

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Abstract

A short introduction to the analytical and algebraic aspects of integrable systems is given. We consider the Riemannian geometry of the isospectral set belonging to the Dirichlet problem $-y'' + q(x)y = \lambda y$, $y(0) = y(1) = 0$, where q is a square integrable function of the real Hilbert space $L^2_{\mathbb{R}}([0,1])$. We derive the metric and the connection for the isospectral set, which is an infinite dimensional real analytic submanifold of $L^2_{\mathbb{R}}([0,1])$, in the case of large eigenvalues. The curvature in the asymptotic case is then derived and it is proved that the connection and the curvature are well defined if we take their coefficients in the discrete Sobolev spaces. We further give the explicit formulae for the parallel transport and a sufficiency condition is derived such that a curve on the isospectral set is a geodesic.

Keywords: Isospectral sets; Riemannian geometry; Asymptotic case
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0. Introduction

This is the first in a series of papers where the geometrical and analytical properties of the isospectral set of the Dirichlet problem

$$-y''(x) + q(x)y(x) = \lambda y(x), \qquad y(0) = y(1) = 0,$$
$$q \in L^2_{\mathbb{R}}[0,1], \quad \lambda \in \mathbb{C}, \quad x \in [0,1]$$

are investigated. First, in this paper, a short introduction to the subject from its analytical and algebraic viewpoint is given, and then the isospectral set is considered as a Riemannian

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manifold for large eigenvalues λ_n of the Dirichlet problem. The second paper deals with the Grassmannian, determinant bundle and the tau function in the asymptotic case. In two further papers a non-asymptotic analysis is performed.

The paper is organized as follows. After the introduction and the review of the facts used from inverse spectral theory, the metric and curvature of the isospectral set are calculated in Section 2 and it is shown that they are well defined if the coefficients are elements of the discrete Sobolev spaces. In Section 3 the operator which transports tangent vectors at the points of the isospectral set parallel to the tangent vectors at new points on the set is explicitly calculated. In Section 4 we consider the question when a curve is a geodesic. The geodesic condition is transformed in an equivalent fixed point equation and a sufficiency condition is given for the existence of a unique solution.

We start by developing the analytical theory of integrable systems in the spirit of Sato [11] (for a more detailed discussion, see the articles of [1,3,8,9,11,14]). Let $W = \sum_{k=0}^{\infty} w_k(x)\partial^{-k}$ be a formal pseudo-differential operator (= ψ DO), where the coefficients lie in some algebra of smooth functions of x. The Leibniz rule implies that the ψ DO has to be shifted to the right in the following way:

$$\partial^{-1}w = \sum_{j=0}^{\infty}(-1)^j(\partial^j w)\partial^{-1-j}, \tag{0.1}$$

which makes the algebra of pseudo-differential operators into an associative algebra. To keep things simple we consider the operators $W_m = \sum_{k=0}^{m} w_k(x)\partial^{-k}$, which keeps the essence of the full theory (see [6,7]). We now consider the ordinary differential equation

$$W_m\partial^m f(x) = 0, \tag{0.2}$$

where we assume that the m solutions $f^{(j)}$, $j = 0, 1, 2, \ldots, m$ are analytic, i.e.

$$f^{(j)}(x) = \sum_{k=0}^{\infty}\xi_k^{(j)}\frac{1}{k!}x^k. \tag{0.3}$$

These m functions $f^{(j)}$ satisfy (0.2), hence, we get a linear $m\times m$ system of equations where we regard the w_j as the unknowns. Using the Cramers rule, w_j and W_m are both expressed as ratios of two determinants involving as matrix entries the functions $f^{(j)}$. Furthermore, if we introduce the $m\times\infty$ matrix K,

$$K = \{\xi_j^{(i)}\}_{1\le i\le m,\quad 0\le j<\infty}, \tag{0.4}$$

then,

$$W_m\partial^m\left(1, x, \frac{x^2}{2!}, \ldots\right)K = 0 \quad\text{and}\quad W_m\partial^m\left(1, x, \frac{x^2}{2!}, \ldots\right)KX = 0$$

holds for any regular matrix X. This implies that

$$K \in Gr := \{\infty\times m \text{ matrices, with rank } m\}/GL(m,\mathbb{C}).$$

Gr is a Grassmann manifold. The energy operator is defined by

$$H(x) := \exp(x\Lambda)K, \tag{0.5}$$

with

$$\Lambda = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix},$$

a shift operator. To attain Sato's equation let $w_j(x)$ depend on an infinite number of variables $w_j(x, t_1, t_2, \dots,)$, where we denote $\boldsymbol{t} = (t_1, t_2, \dots,)$. This implies that the solutions $f^{(j)}(x)$ depend on an infinite set of parameters $\boldsymbol{t}$ and the energy becomes

$$H(x,\boldsymbol{t}) = \exp(x\Lambda)\exp\left(\sum_{n=1}^{\infty} t_n \Lambda^n K\right) = \sum_{n=0}^{\infty} p_n \Lambda^n K, \tag{0.6}$$

where we formally expanded the exponentials and the p_n are the generalized Schur polynomials. Writing $h_j^{(k)}(x,\boldsymbol{t})$, $1 \le k \le j$, $j \in \mathbb{N}$, for the matrix elements of the energy operator and using the property

$$\frac{\partial p_n}{\partial t_m} = p_{n-m} \quad (p_n = 0, n < 0), \tag{0.7}$$

we get that the functions $h_j^{(k)}(x,\boldsymbol{t})$ are the solutions of the PDE

$$\left(\frac{\partial}{\partial t_n} - \frac{\partial^n}{\partial x^n}\right) h_j^{(k)} = 0, \tag{0.8}$$

with the initial conditions

$$h_0^{(k)}(x,0) = f^{(k)}(x).$$

This gives us the analogue equation to (0.2),

$$W_m \partial^m h_0^{(j)}(x,\boldsymbol{t}) = 0, \quad j = 1, 2, \dots, m. \tag{0.9}$$

The time evolution of $W_m = \sum_{k=0}^{\infty} w_k(x,\boldsymbol{t})\partial^{-k}$ is found by differentiating (0.9) w.r.t. t_n, that is, we get Sato's equation

$$\frac{\partial}{\partial t_n} W_m = B_n W_m - W\partial^n, \quad B_n = (W_m \partial^n W_m^{-1})_+, \tag{0.10}$$

where $(A)_+ =$ is the differential operator part of A. From the Sato equation the Lax and the Zakahrov–Shabat equations are easily derived. If we introduce the operator

$$L = W\partial W^{-1} = \sum_{i=0}^{\infty} u_i \partial^{-i+1}, \tag{0.11}$$

then the generalized Lax equation is

$$\frac{\partial L}{\partial t_n} = [B_n, L]. \tag{0.12}$$

If we expand the Lax equation and equate the coefficients in ∂^{-k} we get hierarchies of equations for example the KP hierarchy. To link these analytical facts to a geometric and algebraic viewpoint, we return to the point where we stated that W_m and w_j are expressed as a ratio of two determinants. We have

$$W_m(x,t) = \frac{\sigma(x,t)}{\tau(x,t)} = \frac{\det A}{\det H}, \tag{0.13}$$

where the matrix elements of H are $h_{ij} = h_{j-1}^{(i)}$, $i = 1, 2, \ldots, m$, $j = 0, 1, \ldots, m-1$ and A is the matrix

$$A = \left[\begin{array}{c|c} & \partial^{-m} \\ H & \vdots \\ & \partial^{-1} \\ \hline h_m^{(1)} \ldots h_m^{(m)} & 1 \end{array}\right].$$

Since τ is the determinant of the first m rows of the energy operator, we write τ as the sum of products of determinants,

$$\tau(x,t) = \sum_{S \in \mathcal{S}} \det B_S(t) \det(\xi_S), \tag{0.14}$$

where

$$\begin{aligned} (B_S(t))_{ij} &= p_{l_i - j}, \quad i = 1, \ldots, m, \ j = 0, 1, \ldots, m-1, \\ ((\xi_S))_{ij} &= \xi_{l_j}^{(i)}, \quad i, j = 0, 1, \ldots, m. \end{aligned} \tag{0.15}$$

The sum is over the set $\mathcal{S}$ of all increasing sequences $S = (s_1, s_2, \ldots)$ of integers such that $s_i = i$ for all except a finite number of i. The graphical representation of this set $\mathcal{S}$ are called Maya diagrams which have a one-to-one correspondence with Young diagrams. Any analytic function $f(t)$ can be expanded in the form

$$f(t) = \sum_{S \in \mathcal{S}} \det B_S(t) \det(c_S), \tag{0.16}$$

where c_S are determined by the orthogonality condition

$$\det(c_S) = \det B_S(\tilde{\partial}_t) f(t)|_{t=0}, \quad \tilde{\partial}_t = \left(\frac{\partial}{\partial t_1}, \frac{1}{2}\frac{\partial}{\partial t_2}, \frac{\partial}{\partial t_3}, \ldots\right). \tag{0.17}$$

With the aid of these equations w_j is then expressed in the form

$$w_j = \frac{1}{\tau} p_j(-\tilde{\partial}_t)\tau. \tag{0.18}$$

Since the coefficients w_j of W_m and those of the Lax operator L, u_j are related to each other, the solution function u_2 of the KP equation is written in terms of the tau function in the form

$$u_2 = \frac{\partial^2}{\partial x^2} \log \tau(x, t). \tag{0.19}$$

The coefficients of the tau function have to satisfy the Plücker relations, a fact which indicates that the tau function may be seen as a function acting on a Grassmannian. The Lax equation is the compatibility condition for the linear system

$$L\psi = \lambda\psi, \qquad \frac{\partial \psi}{\partial t_n} = N_n \psi, \qquad \frac{\partial \lambda}{\partial t_n} = 0, \tag{0.20}$$

and the eigenfunctions ψ are given by

$$\psi = \left(1 + \sum_{i=0}^{\infty} w_i(x, t)\lambda^{-i}\right) \exp\left[x + \sum_{i=1}^{\infty} \lambda^i t_i\right]. \tag{0.21}$$

The function ψ is called the Baker function. This is the analytical framework. We now start with the algebraic–geometric construction. Let H be the Hilbert space of square integrable functions on the unit circle with values in $\mathbb{C}$. We split H in the direct sum $H_+ \oplus H_-$, where $H_+(H_-)$ is spanned by $\{z^n, n \geq 0\}$ ($\{z^{-n}, n > 0\}$). The Grassmannian $Gr(H)$ is the set of all closed subspaces W of H, where the orthogonal projection of W into H_+ is a Fredholm operator and the orthogonal projection of W_- into H_- is a compact operator. (We consider the connected component with index zero of the Fredholm operator). The Grassmannian is a Hilbert manifold modeled over the space of the Hilbert–Schmidt operators from W into $W^\perp$. The coordinate charts are indexed by the set S of the Young diagrams. The next construction is the determinant bundle $\mathrm{DET}(W)$ of $Gr(H)$ where an element $q \in \mathrm{DET}(W)$ is represented in the form

$$q = \lambda \bigwedge_{i=0}^{\infty} w_i, \quad \lambda \in \mathbb{C},$$

and $\{w_i\}$ is an admissible basis of W. A basis is admissible if the matrix which describes a coordinate change has a determinant. The group of automorphisms on $\mathrm{DET}(W)$ is the subgroup $GL_1^+(H)$ of the restricted general linear group on H, $GL_{\mathrm{res}}(H)$, where an element $g \in GL_1^+(H)$ has the form

$$g = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \tag{0.22}$$

according to the splitting of H, where a is an invertible operator, and b is the trace class. The group $GL_{\mathrm{res}}(H)$ acts transitively on $Gr(H)$. A specific subgroup of $GL_1^+(H)$ is the group Γ_+ of real analytic functions from the unit circle in $\mathbb{C}^\times$ which extends to a holomorphic function as a map from the unit disc into $\mathbb{C}^\times$ acting on the Hilbert space by multiplication.The tau function is the holomorphic function from Γ_+ to $\mathbb{C}$ defined by

$$\tau_W(g) := \frac{\sigma(g^{-1}W)}{g^{-1}\delta_W} = \det(w_+ + a^{-1}bw_-), \quad g \in \Gamma_+, \tag{0.23}$$

with $w = (w_+, w_-)$ an admissible basis, δ_W a non-zero number in the fiber of the dual bundle $\mathrm{DET}^*(W)$ and σ the canonical global holomorphic section on $\mathrm{DET}^*(W)$. Note that the second equality in (0.23) only holds if g^{-1} has the block form (0.22). Then to each $W \in Gr(H)$ is associated a Baker function

$$\psi_W(g,z) = g(z)\left(1 + \sum_{i=1}^{\infty} w_i(g)z^{-1}\right), \tag{0.24}$$

where $g \in \Gamma_+^W = \{g \in \Gamma_+, g^{-1}W \text{ transverse to } H_-\}$. This Baker function is written as a function of the tau function,

$$\psi_W(g,z) = g(z)\frac{\tau_W(gq_\eta)}{\tau_W(g)} = g(z)\frac{\tau_W(x - \frac{1}{\eta}, t_1 - \frac{1}{2}\eta^2, \ldots)}{\tau_W(x,t)}. \tag{0.25}$$

Comparing this algebraic approach to the analytic one, we see that the two tau functions and the Baker functions are equal.

The problem we consider in this and the forthcoming papers is the Dirichlet problem on [0, 1], i.e. we have other boundary conditions than in the K dV or KP case for example. The Hilbert space will be $L^2_{\mathbb{R}}[0,1]$ and the splitting is according to the even and odd functions. We will see that the elements of the Grassmannian are the tangent spaces of the isospectral sets of the Dirichlet problem. The main difficulty will be the determination of the group acting on the Grassmannian and on the determinant bundle, which is significantly different to the situation discussed above. The fact which makes things more complicated is that the translation symmetry of the circle no longer holds in our case.

1. Facts from inverse spectral theory

The following facts are taken from Ref. [9]. Consider the following Dirichlet problem:

$$\begin{aligned} &-y''(x) + q(x)y(x) = \lambda y(x), \qquad y(0) = y(1) = 0, \\ &\quad q \in L^2_{\mathbb{R}}[0,1], \quad \lambda \in \mathbb{C}, \quad x \in [0,1]. \end{aligned} \tag{1.1}$$

The first result is that the spectrum of q is an infinite sequence of real numbers, which are bounded from below and tends to $+\infty$. Writing $y_2(x,\mu_n) = y_2(x,\mu_n,q)$ for a solution of the differential equation (1.1) where μ_n is the nth eigenvalue of the Dirichlet spectrum of q, we associate a unique eigenfunction $g_n(x,\mu_n)$ to μ_n defined by

$$g_n(x,\mu_n) := \frac{y_2(x,\mu_n)}{\|y_2(\cdot,\mu_n)\|} = \frac{y_2(x,\mu_n)}{\sqrt{\dot{y}_2(1,\mu_n)y_2'(1,\mu_n)}}, \tag{1.2}$$

where "$'$" denotes differentiation w.r.t. x and "$\cdot$" w.r.t. λ, respectively, and we further have $||g_n|| = 1$, $g_n \geq 0$. The gradient of μ_n is given by

$$\frac{\partial \mu_n}{\partial q(t)} = g_n^2(t, q). \tag{1.3}$$

The inverse Dirichlet problem is to answer to what extent a point $p \in L^2$ is determined by its Dirichlet spectrum. Furthermore, given a $q \in L^2$, we ask what does the set of all square integrable functions look like which all have the same Dirichlet spectrum as q, i.e. what kind of set is the isospectral set $M(q) := \{p \in L^2 \mid \mu(q) = \mu(p)\}$. To characterize the sequence of real numbers which arise as the Dirichlet spectrum of some q we look at the function μ in the following way:

$$\mu : L^2 \to S, \qquad q \to \mu(q) = (\mu_1(q), \mu_2(q), \ldots), \tag{1.4}$$

where S is the space of all real, strictly increasing sequences of the form

$$\sigma_n = n^2\pi^2 + s + \tilde{\sigma}_n, \quad n \geq 1 \tag{1.5}$$

with $s \in \mathbb{R}$, $\tilde{\sigma} = (\tilde{\sigma}_1, \tilde{\sigma}_2, \ldots) \in \ell^2$. In order to determine q uniquely from its Dirichlet spectrum we introduce the map $\kappa : L^2 \to \ell_1^2$ which maps q into $\kappa(q) = (\kappa_1(q), \kappa_2(q), \ldots)$, where

$$\kappa_n(q) := \log(-1)^n y_2'(1, \mu_n). \tag{1.6}$$

The numbers κ_n are essentially the terminal velocities. Note that a point q in L^2 is even if and only if $\kappa(q) = 0$. The space ℓ_k^2 is the discrete version of the Sobolev spaces, i.e. it consists of all real valued sequences (a_n) such that $\sum_n (n^k a_n)^2 < \infty$. The basic theorem for the inverse Dirichlet problem is:

Theorem 1 [9, p.116]. *The map $\kappa \times \mu$ from L^2 to $S \times \ell_1^2$ is a real analytic isomorphism.*

We are interested in the asymptotic expansion with respect to large eigenvalues λ_n of the various functions we introduced above.

Theorem 2 [9, pp. 38, 59].

$$\begin{aligned} &g_n(x, q) = \sqrt{2}\sin \pi n x + \mathrm{O}(1/n), && g_n'(x) = \sqrt{2}\pi n \cos \pi n x + \mathrm{O}(1), \\ &\mu_n(q) = n^2\pi^2 + \int_0^1 q(t)\,\mathrm{d}t + \ell^2(n), && g_n^2(x, q) = 1 - \cos 2\pi n x + \mathrm{O}(1/n), \\ &\frac{\mathrm{d}}{\mathrm{d}x} g_n^2(x, q) = 2\pi n \sin 2\pi n x + \mathrm{O}(1), && \kappa_n(q) = \frac{1}{2\pi n}\langle \sin 2\pi n x, q\rangle + \mathrm{O}(1/n^2). \end{aligned} \tag{1.7}$$

All the estimates hold uniformly in bounded subsets of $[0, 1] \times L^2$.

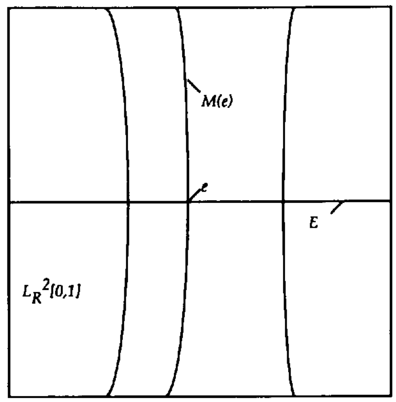

Fig. 1. L^2 as a bundle over E with fibers $M(e)$.

The brackets $\langle\ ,\ \rangle$ in the above theorem denote the usual inner product on the real Hilbert space and $\ell^2(n)$ means that there is a sequence (λ_n) such that $\sum_n |\lambda_n|^2 < \infty$. Since our goal is to do Riemannian geometry on the isospectral set we use the following facts which are all proved in Ch. 4 of [9]. The first is that $M(p)$ is a real analytic submanifold of L^2 lying in the hyperplane of all functions with mean $[p] := \int_0^1 q(t)\,\mathrm{d}t$. Furthermore, L^2 is the direct sum of the normal space $N_q(M(p))$ at the point q and the tangent space $T_q(M(p))$ at the same point. The two spaces are defined by

$$N_q(M(p)) = \{U_\eta(x,q) \mid \eta \in \mathbb{R} \times \ell^2\}, \qquad T_q(M(p)) = \{V_\xi(x,q) \mid \xi \in \ell_1^2\},$$
$$U_\eta(x,q) = \sum_{n\geq 0} \eta_n U_n, \qquad V_\xi(x,q) = \sum_{n\geq 1} \xi_n V_n. \tag{1.8}$$

The components of the two vectors V_ξ and U_η are given by

$$U_0 = 1, \qquad U_n = g_n^2 - 1, \qquad V_n = 2\frac{\mathrm{d}}{\mathrm{d}x} g_n^2. \tag{1.9}$$

There is another characterization for L^2 which is very suggestive for our geometric considerations, i.e. denoting the set of even square integrable functions by E and the set of odd ones by U, we can write $L^2 = E \oplus U$. $M(e)$ intersects E in exactly one even point e and nowhere else (Fig. 1). Therefore, we have

$$L^2 = \bigcup_{e\in E} M(e). \tag{1.10}$$

The solution curves $\Phi^t(q) := \Phi^t(q, V_\eta)$ of V_η are given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi^t(q) = V_\eta(\Phi^t(q)), \quad a < t < b, \ \ \Phi^0(q) = q, \tag{1.11}$$

where we assume that $a < 0 < b$ and that the curve is C^∞.

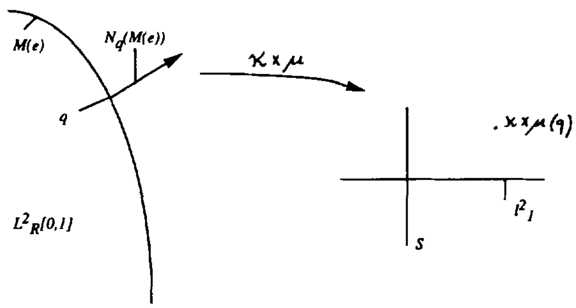

Fig. 2. Coordinate chart map from L^2 into S.

2. The metric, connection and curvature of $M(e)$ in the asymptotic case

Let $q \in M(e)$. Then the asymptotic metric components $\hat{g}_{jk}(q)$, or just the metric components, at the point q are defined by

$$\hat{g}_{jk}(q) = \int_0^1 V_j(x,q)V_k(x,q)\xi_j\eta_k \, \mathrm{d}x, \tag{2.1}$$

where the sequences (ξ_n) and (η_n) are elements of ℓ_1^2 and $V_j, V_k \in T_q(M(e))$. We have:

Lemma 1. *The asymptotic metric components are given by*

$$\hat{g}_{jk}(q) = 8\pi jk\delta_{jk} + \mathrm{O}(1), \qquad \sum_{j,k=1}^{\infty} \hat{g}_{jk}(q)\xi_j\eta_k < \infty. \tag{2.2}$$

Proof. We insert the asymptotic expression for the tangent vectors (1.8) and write

$$\begin{aligned}\hat{g}_{jk}(q) &= 4\int_0^1 \frac{\mathrm{d}}{\mathrm{d}x}g_j^2(x,q)\frac{\mathrm{d}}{\mathrm{d}x}g_k^2(x,q)\,\mathrm{d}x \\ &= 4\int_0^1 \Big(4\pi j \sin 2\pi jx + \mathrm{O}(1)\Big)\Big(4\pi k \sin 2\pi kx + \mathrm{O}(1)\Big)\,\mathrm{d}x \\ &= 8\pi jk\delta_{jk} + \mathrm{O}(1),\end{aligned}$$

where we used the notation $g_j^2(x,q) := g_j^2(x,q,\mu_j(q))$. The claim that $\sum_{j,k=1}^{\infty} \hat{g}_{jk}(q)\xi_j\eta_k < \infty$ is clear. □

The next task is the calculation of the asymptotic connection and the asymptotic curvature. Fig. 2 shows the geometric setting.

We prove the following lemma.

Lemma 2. *The connection of the isospectral set for large eigenvalues and for $\xi_j, \eta_k, \sigma_i \in \ell^2$ is given by*

$$\Gamma_{ij}^{k} = \sum_{l=0}^{\infty} \mathrm{O}\left(\frac{i^2+j^2+l^2}{kl(i+j+l)}\right)\xi_j \eta_k \sigma_i \nu_l,$$

which is well defined if $\nu_l \in \ell_1^2$.

Proof. Let $\kappa(q) = u \in \ell_1^2$ and V_j, V_k be elements of the tangent space at $q \in M(e)$. To determine the connection we need to calculate terms of the form

$$\frac{\partial}{\partial u_i}\hat{g}_{jk}(u). \tag{2.3}$$

Using the representation

$$V_j = 2\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial}{\partial q}\mu_j(q) \tag{2.4}$$

for the tangent vectors we have

$$\begin{aligned}\frac{\partial}{\partial u_i}\hat{g}_{jk}(u) &= \frac{\partial}{\partial u_i}\int_0^1 V_j(x,\kappa^{-1}(u))V_k(x,\kappa^{-1}(u))\xi_j\eta_k \,\mathrm{d}x \\ &= 8\int_0^1 \Big(\frac{\mathrm{d}}{\mathrm{d}x}(\partial_{u_i}\kappa^{-1}(u)g_j\partial_{\kappa^{-1}(u)}g_j)V_k \\ &\quad + \frac{\mathrm{d}}{\mathrm{d}x}(\partial_{u_i}\kappa^{-1}(u)g_k\partial_{\kappa^{-1}(u)}g_k)V_j\Big)\xi_j\eta_k\,\mathrm{d}x.\end{aligned} \tag{2.5}$$

Since the estimates for V_j hold uniformally on bounded subsets of $[0,1]\times L^2$ we are allowed to interchange integration and differentiation. To justify the change of order of the derivatives w.r.t. x and u_i, first suppose that in the above equation μ_j is twice differentiable in q. Since both sides of the equation hold in a dense subset of L^2, this also holds in general by continuity in q. We start with the calculation of $2g_j\kappa^{-1}(u)g_j = (\partial^2/\partial(\kappa^{-1})^2)\mu_j$. Writing for simplicity $q := \kappa^{-1}(u)$, we have for the derivative of g_j:

$$\frac{\partial}{\partial q}g_n = \frac{\partial_q y_2}{\sqrt{\dot{y}_2(1,\mu_n,q)y_2'(1,\mu_n,q)}} - \frac{1}{2}g_n\partial_q \log \dot{y}_2(1,\mu_n,q)y_2'(1,\mu_n,q). \tag{2.6}$$

With this formula we can write the second derivative of μ_n w.r.t. q as

$$\begin{aligned}\frac{\partial^2}{\partial q^2}\mu_n = &-g_n^2\partial_q \log(\dot{y}_2(1,\mu_n,q)y_2'(1,\mu_n,q)) \\ &+ \frac{2g_n}{\sqrt{\dot{y}_2(1,\mu_n,q)y_2'(1,\mu_n,q)}}\partial_q y_2.\end{aligned} \tag{2.7}$$

Using the asymptotic expansion of y_2 given by (1.7), we have

$$\partial_q \dot{y}_2(1,\mu_n,q)y_2'(1,\mu_n,q)$$
$$= \int_0^1 \Big(\frac{\cos\sqrt{\mu_n}t \sin\sqrt{\mu_n}t}{2\mu_n\sqrt{\mu_n}} - \frac{\sin^2\sqrt{\mu_n}t}{\mu_n^2}\Big)\frac{\partial\mu_n}{dq}\,\mathrm{d}t + \mathrm{O}\left(\frac{1}{n^3}\right). \tag{2.8}$$

Since $\int_0^1 \cos\sqrt{\mu_n}t \sin\sqrt{\mu_n}t\,\mathrm{d}t = (1/n)$ and $\int_0^1 \sin^2\sqrt{\mu_n}t\,\mathrm{d}t = (1/2\mu_n)(1+\mathrm{O}(1))$ we get

$$\partial_q \dot{y}_2(1,\mu_n,q)y_2'(1,\mu_n,q) = \frac{1}{\mu_n^2}\left(\sqrt{\mu_n}\mathrm{O}\left(\frac{1}{n}\right) - \frac{1}{2} + \mathrm{O}\left(\frac{1}{n}\right)\right). \tag{2.9}$$

Using the asymptotic expansion of g_n^2 (see (1.7)) we can finally write for the first term in (2.7)

$$-g_n^2\partial_q \log(\dot{y}_2(1,\mu_n,q)y_2'(1,\mu_n,q)) = \mathrm{O}(1/n). \tag{2.10}$$

The next term in (2.7) we have to deal with is the derivative of y_2 w.r.t. q. Using the asymptotic expansion of (1.7) we calculate

$$\partial_q y_2 = \partial_q\Big(\frac{\sin\sqrt{\mu_n}t}{\mu_n}\Big) = \frac{\partial\mu_n}{\partial q}\frac{\partial}{\partial\mu_n}\Big(\frac{\sin\sqrt{\mu_n}t}{\mu_n}\Big)$$
$$= \mathrm{O}(1/n)g_n^2 = \mathrm{O}(1/n)(1-\cos 2\pi n x + \mathrm{O}(1/n)) = \mathrm{O}(1/n). \tag{2.11}$$

Therefore, we get the following expression for the second term in (2.7):

$$\frac{2g_n}{\sqrt{\dot{y}_2(1,\mu_n,q)y_2'(1,\mu_n,q)}}\partial_q y_2$$
$$= \mathrm{O}(1/n)(\pi n + \mathrm{O}(1))(1+\mathrm{O}(1/n))(\sin\sqrt{\mu_n}x + \mathrm{O}(1/n)) = \mathrm{O}(1). \tag{2.12}$$

(2.10) and (2.12) together prove that

$$\frac{\partial^2}{\partial q^2}\mu_n = \mathrm{O}(1). \tag{2.13}$$

With this result we continue to calculate $\partial_{u_i}\hat{g}_{jk}(u)$.

$$\partial_{u_i}\hat{g}_{jk}(u) = 8\mathrm{O}(1)\int_0^1 \Big(\frac{\mathrm{d}}{\mathrm{d}x}(\partial_{u_i}\kappa^{-1}(u))V_k + \frac{\mathrm{d}}{\mathrm{d}x}(\partial_{u_i}\kappa^{-1})V_j\Big)\xi_j\eta_k\,\mathrm{d}x.$$

Since

$$\partial_{u_i}\kappa^{-1}(u) = \sigma_i V_i \kappa^{-1}(u), \quad (\sigma_n)_{n\in\mathbb{N}} \in \ell_1^2, \tag{2.14}$$

we get

$$\frac{\mathrm{d}}{\mathrm{d}x}(\partial_{u_i}\kappa^{-1})(u) = 4\pi i \sin 2\pi i x + \mathrm{O}(1).$$

Hence

$$(2.13) = \mathrm{O}(1)\int_0^1 \left(\left(\frac{\mathrm{d}}{\mathrm{d}x}V_i\right)V_k + \left(\frac{\mathrm{d}}{\mathrm{d}x}V_i\right)V_j\right)\xi_j\eta_k\sigma_i\,\mathrm{d}x$$
$$= \mathrm{O}(i^2)\Big(-\frac{\cos(2\pi i + 2\pi k)x}{2(2\pi i + 2\pi k)} - \frac{\cos(2\pi i - 2\pi k)x}{2(2\pi i - 2\pi k)}\Big\|_0^1(1-\delta_{ik})\Big)\xi_j\eta_k\sigma_i$$
$$= \mathrm{O}\left(\frac{i^2}{i+k+j}\right)\xi_j\eta_k\sigma_i. \tag{2.15}$$

In order that the sum over i, j, k of (2.15) is convergent we have to assume that $\sigma_i \in \ell_1^1$ and that $\xi_j, \eta_k \in \ell^2$. Therefore,

$$\frac{\partial\hat{g}_{lj}}{\partial\hat{u}^i} + \frac{\partial\hat{g}_{il}}{\partial\hat{u}^j} - \frac{\partial\hat{g}_{ij}}{\partial\hat{u}^l} = \mathrm{O}\left(\frac{i^2+j^2+k^2}{i+j+k}\right)\xi_j\eta_k\sigma_i. \tag{2.16}$$

To obtain the connection form, we have to invert the metric,

$$\hat{g}^{kl} = (8\pi\delta_{kl} + \mathrm{O}(1))^{-1}$$
$$= \frac{1}{8\pi}\delta_{kl}\left(1 + \mathrm{O}(1)\frac{1}{8\pi}\delta_{kl}\right)^{-1} = \frac{1}{8\pi}\delta_{kl} + \mathrm{O}\left(\frac{1}{k^2l^2}\right). \tag{2.17}$$

This implies for the connection the following expression for large eigenvalues

$$\Gamma_{ij}^k = \sum_{l=0}^{\infty}\frac{1}{2}\left\{\hat{g}^{kl}\Big(\frac{\partial\hat{g}_{lj}}{\partial\hat{u}^i} + \frac{\partial\hat{g}_{il}}{\partial\hat{u}^j} - \frac{\partial\hat{g}_{ij}}{\partial\hat{u}^l}\Big)\nu_l\right\}$$
$$= \sum_{l=0}^{\infty}\mathrm{O}\left(\frac{i^2+j^2+l^2}{kl(i+j+l)}\right)\xi_j\eta_k\sigma_i\nu_l. \tag{2.18}$$

This proves Lemma 2. □

The same kind of calculations gives us for the curvature the following expression:

Theorem 3. *The asymptotic curvature on the isospectral set for*

$$R_{kji}^h = \sum_{l=0}^{\infty}(\partial_k\Gamma_{ji}^h - \partial_j\Gamma_{ki}^h + \Gamma_{kl}^h\Gamma_{ji}^l - \Gamma_{jl}^h\Gamma_{ki}^l)\xi_j\eta_k\sigma_i\nu_l$$

has for $\xi_j, \eta_k, \sigma_i \in \ell^2$ *the form*

$$R_{kji}^h = \sum_{l=0}^{\infty}\mathrm{O}\left(\frac{(h^2+k^2+l^2)(l^2+j^2+i^2)}{h^2l^2(h+k+l)(i+j+l)}\right)\xi_j\eta_k\sigma_i\nu_l$$
$$+\sum_{l=0}^{\infty}\mathrm{O}\left(\frac{(h^2+j^2+l^2)(l^2+j^2+i^2)}{h^2l^2(h+j+l)(i+k+l)}\right)\xi_j\eta_k\sigma_i\nu_l, \tag{2.19}$$

which is well defined if $\nu_l \in \ell_2^2$.

3. The parallel transport in the asymptotic case

Let c be a curve on the isospectral set whose components satisfy the differential equation

$$\dot{c}^t := \frac{\mathrm{d}c^t(q)}{\mathrm{d}t} = V(c^t(q)), \quad c^{0,k}(q) = q, \; a < t < b, \tag{3.1}$$

and we assume that the curve is smooth. Using $V_k = 4\pi k \sin 2\pi kx + \mathrm{O}(1)$ and (2.18) we have

$$\begin{aligned} \Gamma^i_{jk}\dot{c}^{t,k}\xi_j\eta_i\sigma^k &= \sum_{l=0}^{\infty} \mathrm{O}\left(\frac{k^2+j^2+l^2}{il(k+j+l)}\right)(4\pi k \sin 2\pi kx + \mathrm{O}(1))\xi_j\eta_i\nu_l\sigma^k \\ &=: (4\pi k \sin 2\pi kx + \mathrm{O}(1))F^i_{jk}, \end{aligned} \tag{3.2}$$

where $\sigma_k \in \ell^2_1$.

Theorem 4.

(a) *Let $q \in E$. Then the tangent vector $U(x, c^t(q))$ is reached from the tangent vector $U(x,q)$ along an integral curve $c^t(q, V_\sigma)$, which satisfies $\dot{c}^t = U(c^t(q))$, $c^0(q) = q$, $a < t < b$, by the transformation*

$$\begin{aligned} &U^i(x, c^t(q(x))) \\ &\quad= \sum_{j,k>0}\left(\exp\left\{-\sum_{l=0}^{\infty}(4\pi k \sin 2\pi kx + 1)\mathrm{O}\left(\frac{k^2+j^2+l^2}{il(k+j+l)}\right)\right.\right. \\ &\qquad \left.\left. \times \int_0^t \exp\left[\sum_{n=1}^{\infty}\delta_n \sinh^2(\tfrac{1}{2}s\xi_n)\mathrm{d}s\xi_j\eta_i\nu_l\sigma^k\right]\right\}\right) U^j(x, q(x)), \end{aligned}$$

where $\xi_j, \eta_k, \sigma_i \in \ell^2$, $\nu_l \in \ell^2_2$.

(b) *Let $q \in E$. Then the tangent vector $U(x, c^t(q))$ is reached from the tangent vector $U(x,q)$ along an integral curve $c^t(q, V_1)$, which satisfies $\dot{c}^t = U(c^t(q)), c^0(q) = q$, $a < t < b$ and $V_1 := (1, 0, 0, \ldots)$, by the transformation*

$$\begin{aligned} &U^i(x, c^t(q(x))) \\ &\quad= \sum_{j,k>0}\left(\exp\left\{-\sum_{l=0}^{\infty}(4\pi k \sin 2\pi kx + 1)\mathrm{O}\left(\frac{k^2+j^2+l^2}{il(k+j+l)}\right)\xi_j\eta_i\nu_l\sigma^k\right.\right. \\ &\qquad \left.\left. \times(-1)^p\binom{2p}{p}\frac{1}{2^{2p}}t + \frac{1}{2^{2p}}\sum_{k=0}^{p-1}(-1)^k\binom{2p}{k}\frac{\sinh(2p-2k)t}{2p-2k}\right\}\right) U^j(x, q(x)), \end{aligned}$$

where $\xi_j, \eta_k, \sigma_i \in \ell^2$, $\nu_l \in \ell^2_2$.

Proof. First we prove (a): The parallel transport equation we consider is

$$\frac{\mathrm{d}U^i(x, c^t(q(x)))}{\mathrm{d}t} = -\dot{c}^{t,k}\Gamma^i_{jk}U^j(x, q(x)), \quad U^i(t=0) = U^i(x, q(x)), \tag{3.3}$$

where $U^i(x, q(x))$ is a component of the tangent vector $U_\eta \in N_q(M(e))$. The solution of (3.3) is

$$U^i(x, c^t(q(x))) = \sum_{j,k>0} \left(T \left[\exp\left\{ - \int_0^t \dot{c}^{s,k} \Gamma^i_{jk} \, ds \right\} \right] \right)_{ji} U^j(x, q(x)), \tag{3.4}$$

where the operator T denotes that the integral is a time ordered integral. It is defined in the following way: Let $A(t)$ be a bounded operator on a Hilbert space, then

$$\begin{aligned} T & \left[\exp\left\{ - \int_0^t A(s) \, ds \right\} \right] \\ & := \sum_{n=0}^{\infty} \sum_{\mathcal{S}_n} \frac{(-1)^n}{n!} \int_0^t dt_1 \int_0^t dt_2 \cdots \int_0^t dt_n \Theta_n(t_{\sigma_1}, \ldots, t_{\sigma_n}) A_{\sigma_1}(t_{\sigma_1}) \ldots A_{\sigma_n}(t_{\sigma_n}) \\ & = \sum_{n=0}^{\infty} (-1)^n \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n \Theta_n(t_{\sigma_1}, \ldots, t_{\sigma_n}) A_{\sigma_1}(t_{\sigma_1}) \ldots A_{\sigma_n}(t_{\sigma_n}) \end{aligned} \tag{3.5}$$

with $\mathcal{S}_n$ the symmetric group of n elements, $(\sigma_1, \ldots, \sigma_n)$ a permutation of n elements and the function $\Theta_n(t_{\sigma_1}, \ldots, t_{\sigma_n})$ is equal to one if $t_{\sigma_1} \le t_{\sigma_2} \le \cdots \le t_{\sigma_n}$ and zero else. We start calculating (3.4):

$$\begin{aligned} & \int_0^t \dot{c}^{s,k} \Gamma^i_{jk} \, ds \, \xi_j \eta_i \sigma^k \\ & = \int_0^t \sum_{l=0}^{\infty} O\left(\frac{k^2 + j^2 + l^2}{il(k + j + l)} \right) (4\pi k \sin 2\pi kx + O(1)) \xi_j \eta_i \nu_l \sigma^k \\ & = \sum_{l=0}^{\infty} (4\pi k \sin 2\pi kx + O(1)) O\left(\frac{k^2 + j^2 + l^2}{il(k + j + l)} \right) \int_0^t e^{\|c^s(q)\|_2^2} \, ds \, \xi_j \eta_i \nu_l \sigma^k, \end{aligned} \tag{3.6}$$

where we used (1.6). The norm $\|c^s(q)\|_2^2$ has the expression (see [9])

$$\|c^s(q)\|_2^2 = \|q\|_2^2 + 8 \sum_{n \ge 1} \delta_n (\cosh(\kappa_n(q) + s\xi_n) - \cosh \kappa_n(q)), \tag{3.7}$$

with $\delta_n = 2\pi^2 n^2 (1 + O(\log n/n))$. We restrict for simplicity to the case where $q \in E$. Using that $\kappa(q) = 0$, $\forall q \in E$ and the identity $2 \sinh^2 x/2 = \cosh x - 1$ we obtain

$$\|c^s(q)\|_2^2 = \|q\|_2^2 + 16 \sum_{n \ge 1} \delta_n \sinh^2(\tfrac{1}{2} s \xi_n). \tag{3.8}$$

Inserting (3.8) into (3.6) gives

$$\int_0^t \dot{c}^{s,k} \Gamma^i_{jk} \, \mathrm{d}s \, \xi_j \eta_i \sigma^k$$
$$= \sum_{l=0}^{\infty} (4\pi k \sin 2\pi k x + 1) \mathrm{O}\left(\frac{k^2 + j^2 + l^2}{il(k+j+l)} \right)$$
$$\times \int_0^t \exp\left\{ \sum_{n=1}^{\infty} \delta_n \sinh^2(\tfrac{1}{2} s \xi_n) \, \mathrm{d}s \, \xi_j \eta_i \nu_l \sigma^k \right\}. \tag{3.9}$$

We now prove part (b). We then have

$$\int_0^t \dot{c}^{s,k} \Gamma^i_{jk} \, \mathrm{d}s \, \xi_j \eta_i \sigma^k$$
$$= \sum_{l=0}^{\infty} \mathrm{O}\left(k \frac{k^2 + j^2 + l^2}{il(k+j+l)} \right) \sum_{p=0}^{\infty} \frac{(2\pi^2)^p}{p!} \int_0^t \sinh^{2p}(\tfrac{1}{2} s \xi_n) \, \mathrm{d}s \, \xi_j \eta_i \nu_l \sigma^k.$$

Since

$$\int \sinh^{2m} x \, \mathrm{d}x = (-1)^m \binom{2m}{m} \frac{1}{2^{2m}} x + \frac{1}{2^{2m-1}} \sum_{k=0}^{m-1} (-1)^k \binom{2m}{k} \frac{\sinh(2m-2k)x}{2m-2k},$$

we get

$$\int_0^t \dot{c}^{s,k} \Gamma^i_{jk} \, \mathrm{d}s \, \xi_j \eta_i \sigma^k$$
$$= \sum_{l=0}^{\infty} \mathrm{O}\left(k \frac{k^2 + j^2 + l^2}{il(k+j+l)} \right) \sum_{p=0}^{\infty} \frac{(2\pi^2)^p}{p!}$$
$$\times \left((-1)^p \binom{2p}{p} \frac{1}{2^{2p}} t + \frac{1}{2^{2p}} \sum_{k=0}^{p-1} (-1)^k \binom{2p}{k} \frac{\sinh(2p-2k)t}{2p-2k} \right) \xi_j \eta_i \nu_l \sigma^k. \tag{3.10}$$

Since the time matrices appearing in the time ordered product (3.4) are diagonal and hence commute, we can avoid the time ordering operator in this asymptotic case and we get

$$U^i(x, c^t(q(x)))$$
$$= \sum_{j,k>0} \left(\exp\left\{ - \sum_{l=0}^{\infty} \mathrm{O}\left(k \frac{k^2 + j^2 + l^2}{il(k+j+l)} \right) \xi_j \eta_i \nu_l \sigma^k \right.\right.$$
$$\left.\left. \times \left((-1)^p \binom{2p}{p} \frac{1}{2^{2p}} t + \frac{1}{2^{2p}} \sum_{k=0}^{p-1} (-1)^k \binom{2p}{k} \frac{\sinh(2p-2k)t}{2p-2k} \right) \right\} \right)$$
$$\times U^j(x, q(x)). \tag{3.11}$$

This is the desired explicit expression for the parallel transport in the case of the example. □

Note that the operator in (3.11) acting on the tangent vector component is only a bounded operator if we restrict the time t to be finite. Therefore, we cannot reach a vector $V \in N_{c^\infty(e)}(M(e))$ by parallel transport from a tangent vector lying in the even space E.

4. Geodesics on the isospectral set

The goal in this section is to select out the geodesics from the curves on the isospectral set. Let $c^s : [a,b] \to M(e)$ be a C^∞ curve on M where s denotes the arc length of c, i.e. $||\dot{c}||_2 = 1$. A variation of the curve c^s is defined by the smooth mapping $\phi : [a,b] \times (-\epsilon,\epsilon) \to M(e)$, $\epsilon \in \mathbb{R}$ which satisfies:

(a) $\phi(s,0) = C^s$, $\forall s \in [a,b]$, and

(b) there exists a subdivision $[a = t_0, t_1, \ldots, t_k = b]$ such that $\phi(s,v) =: \phi^{s,v}, v \in (-\epsilon,\epsilon)$ is a C^∞ mapping on each partition $[t_{l-1}, t_l] \times (-\epsilon,\epsilon)$ for $l = 1,2,\ldots,k$.

The tangent space $T_q(M(e))$ at $q \in M(e)$ consists of all vectors $V_\xi = \sum_{n\geq 0} \xi_n V_n(x,q)$, $(\xi_n) \in \ell_1^2$, where $V_n(x,q) = 2(\mathrm{d}/\mathrm{d}x)g_n^2(x,q)$. The tangent vector for each fixed time t of $\phi^{s,v}$ is given by $\partial\phi^{s,v}/\partial v$ and the vector field W along c^s defined by

$$W(x,q) = \left(\frac{\partial \phi^{s,v}}{\partial v}\right)\bigg|_{v=0} \tag{4.1}$$

is called the infinitesimal variation induced from c^s. The first task is to calculate the covariant derivative of a tangent vector in $T_q(M(e))$ w.r.t. another tangent vector given the connection (2.18). Let $A_\beta, B_\sigma \in T_q(M(e))$, then the covariant derivative of A_β in the direction B_σ is defined by

$$\nabla_{B_\sigma} A_\beta = \lim_{t\to 0} \frac{1}{t}[(\tau^t)^{-1} A_{t\beta} - A_\beta], \tag{4.2}$$

where τ^t is the integral curve of B_σ starting at $q \in E$ and

$$A_{t\beta}(q) = \sum_n \beta_n U_n(x, c^t(q)), \quad \{\beta_n\}_{n\in\mathbb{N}} \in \ell_1^2.$$

Lemma 3. *Let $A_\beta, B_\sigma \in T_q(E)$, then the covariant derivative of A_β in the direction B_σ is given by*

$$\nabla_{B_\sigma} A_\beta = \sum_n \left(\sum_{j,k}\sum_l \mathcal{O}_{j,k,l,n}\right)\beta_n U_n(x,q), \tag{4.3}$$

where

$$\mathcal{O}_{j,k,l,n} = \mathrm{O}\left(k\frac{k^2+j^2+l^2}{nl(k+j+l)}\right)\xi_j \nu_l \sigma^k, \tag{4.4}$$

$$\sigma_k \in \ell_2^2, \quad \xi_j, \nu_l \in \ell_1^2, \quad \xi_n \in \ell^2, \quad \beta_m \in \ell_1^2.$$

Proof. From (3.9) we get

$$\begin{aligned}(\tau^t)^{-1} A_{t\beta} - A_\beta \\ = \sum_{n>0} \left(\sum_{j,k>0} \prod_{l\geq 1} \left(\exp\left\{ \mathcal{O}_{j,k,l,n} \int_0^t \exp\left[\sum_{m=1}^\infty \delta_m \sinh^2(\tfrac{1}{2} s \xi_m) \right] \mathrm{d}s \right\} s \right) - 1 \right) \\ \times \beta_n U^n(x, q(x)).\end{aligned}$$

Dividing by t and taking the limit $t \to 0$, which is the same as to derive

$$\int_0^t \exp\left\{ \sum_{m=1}^\infty \delta_m \sinh^2(\tfrac{1}{2} s \xi_m)\, \mathrm{d}s \right\}$$

at $t = 0$, gives us the desired result. □

To get a necessary and sufficient condition for a curve to be a geodesic we use the following theorem.

Theorem 5. *Let $c^t : [a,b] \to M(e), ||\dot{c}|| = 1$. The curve c^t is a geodesic if for every variation $\phi : [a,b] \times (-\epsilon, \epsilon) \to M(e)$ of c^t the equality*

$$\int_a^b g(\nabla_{\dot{c}} \dot{c}, W)\, \mathrm{d}s = 0 \tag{4.5}$$

holds, where W is defined in (4.1).

Before we start checking the implications of this theorem in our situation, we first give the parallel transport equation and the formula for the covariant derivative in the case where the point $q \in Ł^2_{\mathbb{R}}[0, 1]$ is not lying in the subspace of the even functions E. The parallel transport equation reads

$$\begin{aligned}U^i(x, c^t(q(x))) \\ = \sum_{j,k>0} \prod_{l\geq 1} \left(\exp\left\{ - \mathcal{O}_{j,k,i,} \right.\right. \\ \left.\left. \times \int_s^t \exp\left[\sum_{n=1}^\infty \delta_n [\cosh(\kappa_n(2s\xi_n)) - \cosh(s\xi_n)] \right] \mathrm{d}s \right\} \right) U^j(x, c^s(q(x))),\end{aligned} \tag{4.6}$$

where we used that $\kappa_n(c^s(q)) = \kappa_n(q) + s\xi_n$ in the κ-coordinate system. The covariant derivative of $A_\beta(x, c^t(q))$ in the direction B_σ along the integral curve c^t starting at $c^s(q)$ is given by

$$\nabla_{B_\sigma} A_\beta = \sum_n \left(\sum_{j,k} \sum_l \mathcal{O}_{j,k,l,n} \exp\left\{ \sum_{m\geq 1} \delta_m h_m(s) \right\} \right) \beta_n U_n(c^s(q), x), \tag{4.7}$$

where

$$h_m(s) = \cosh(\kappa_n(2s\xi_n)) - \cosh(s\xi_n).$$

The strategy to prove a sufficient condition for a curve on the isospectral set to be a geodesic is first to write out condition (4.5) in our situation, second to express this result in an equivalent fixed point equation and third to constrain the functions appearing in the fixed point equation in a way that we get a unique solution of this equation.

Lemma 4.

$$\begin{aligned}
&\int_a^b g(\nabla_{\dot c}\dot c, W)\,\mathrm{d}s \\
&= \sum_i \beta_i \mu_i \sum_{j,k} \sum_l \mathcal{O}_{j,k,l,i} \\
&\times \left(8 \int_a^b c^s(q) \exp\left\{ \sum_{m\geq 1} \delta_m [\cosh(\kappa_m(2s\xi_m)) - \cosh(s\xi_m)] \right\} \right. \\
&\left. \times (64\pi^3 i^4 \sin^2(2i\pi x) + \mathrm{O}(i^3))\,\mathrm{d}s \right) + \sum_{ih} \beta_i \mu_j \sum_{j,k} \sum_l \left(\mathcal{O}_{j,k,l,i} \mathrm{O}(ih) \right. \\
&\left. \times \int_a^b c^s(q) \exp\left\{ \sum_{m\geq 1} \delta_m [\cosh(\kappa_m(2s\xi_m)) - \cosh(s\xi_m)] \right\} \mathrm{d}s \right) = 0,
\end{aligned}$$

where $\mathcal{O}_{j,k,l,i}$ is given in (4.4), *W is the infinitesimal variation defined in* (4.1) *and c^t : $[a,b] \to M(e), ||\dot c|| = 1$.*

Proof. Rewriting the metric in the form

$$g(\nabla_{\dot c}\dot c, W) = \sum_{ih} g_{ih} (\nabla_{\dot c}\dot c)^i W^h = \sum_{ih} (8\pi i h \delta_{ih} + \mathrm{O}(1))(\nabla_{\dot c}\dot c)^i W^h,$$

we get

$$\begin{aligned}
&\int_a^b g(\nabla_{\dot c}\dot c, W)\,\mathrm{d}s \\
&= \sum_{ih} \int_a^b (8\pi i h \delta_{ih} + \mathrm{O}(1))(\nabla_{\dot c}\dot c)^i W^h\,\mathrm{d}s \\
&= \sum_{ih} \left(8 \int_a^b \phi(s,0) \pi i h \delta_{ih} (\nabla_{\dot c}\dot c)^i T_\mu^h\,\mathrm{d}s + \mathrm{O}(1) \int_a^b \phi(s,0)(\nabla_{\dot c}\dot c)^i T_\mu^h\,\mathrm{d}s \right),
\end{aligned}$$

where we introduced the variation of the path c^t explicitly, i.e.

$$\phi(s, v) = \exp_{c^s(q)}(V_\xi + vT_\mu).$$

Going on with the calculations and setting $\phi(s, 0) = c^s(q)$, we get

$$
\begin{aligned}
&= \sum_{ih} \left(8 \int_a^b c^s(q)\pi ih\delta_{ih} (\nabla_{\dot c}\dot c)^i T_\mu^h \, ds + O(1) \int_a^b c^s(q) (\nabla_{\dot c}\dot c)^i T_\mu^h \, ds \right) \\
&= \sum_{ih} \beta_i \sum_{j,k} \sum_l \mathcal{O}_{j,k,l,i} \left(8 \int_a^b c^s(q)\pi ih\delta_{ih} \exp\left\{ \sum_{m\geq 1} \delta_m h_m(s) \right\} U_i(c^s(q)) T_\mu^h \, ds \right. \\
&\quad \left. + O(1) \int_a^b c^s(q) \exp\left\{ \sum_{m\geq 1} \delta_m h_m(s) \right\} U_i(c^s(q)) T_\mu^h \, ds \right) \\
&= \sum_i \beta_i \mu_i \sum_{j,k} \sum_l \mathcal{O}_{j,k,l,i} \left(8 \int_a^b c^s(q) \exp\left\{ \sum_{m\geq 1} \delta_m h_m(s) \right\} \right. \\
&\quad \left. \times (64\pi^3 i^4 \sin^2(2i\pi x) + O(i^3)) \, ds \right) \\
&\quad + \sum_{ih} \beta_i \mu_j \sum_{j,k} \sum_l \left(\mathcal{O}_{j,k,l,i} O(ih) \int_a^b c^s(q) \exp\left\{ \sum_{m\geq 1} \delta_m h_m(s) \right\} ds \right). \qquad (4.8)
\end{aligned}
$$

This proves the lemma. □

To find an explicit solution of (4.8) is hopeless. However, we apply the Banach fixed point theorem to show at least existence and uniqueness of a solution for a sufficiently small time interval $[a, b]$ and for to be specified bounds on q. Using (see [9])

$$
\begin{aligned}
&c^t(q, V_\xi) = q - 2\frac{d^2}{dx^2} \log\det \Theta(x, t\xi, q), \\
&\Theta(x, t\xi, q)_{ij} = \delta_{ij} + (e^{t\xi_i} - 1) \int_0^x g_i(w, q(w), \lambda_i) g_j(w, q(w), \lambda_i) \, dw, \\
&\{\xi_n\}_{n\in\mathbb{N}} \in \ell_1^2, \qquad (4.9)
\end{aligned}
$$

we can write (4.8) in the equivalent form

$$q \int_a^b I(s) \, ds = \int_a^b I(s) \Xi(x, s, q) \, ds, \qquad (4.10)$$

where

$$I(s) = \exp\left\{ \sum_{m\geq 1} \delta_m[\cosh(\kappa_m(2s\xi_m)) - \cosh(s\xi_m)] \right\}$$

$$\Xi(x,s,q) = 2\frac{\mathrm{d}^2}{\mathrm{d}x^2} \log\det\left\{ \delta_{ij} + (\mathrm{e}^{t\xi_i} - 1) \int_0^x g_i(w,q(w),\lambda_i) g_j(w,q(w),\lambda_i)\,\mathrm{d}w \right\}. \tag{4.11}$$

We write (4.10) in the fixed point equation form

$$q = G(a,b,q,\xi). \tag{4.12}$$

Theorem 6. *The fixed point equation* (4.12) *has a unique solution if*

$$|(1-\mathrm{e}^{t\xi_j})| \int_0^1 |g_i|^4 \int_0^1 |g_j|^4 < 1, \quad \forall i,j,$$

$$\frac{\mathrm{O}(j)}{1 - |(1-\mathrm{e}^{t\xi_j})| \int_0^1 |g_i|^4 \int_0^1 |g_j|^4} \leq \mathrm{O}\left(\frac{\mathrm{e}^t}{j^2}\right), \quad \forall i,$$

$$|\mathrm{e}^{t\xi_i} - 1|\mathrm{O}(i) < \mathrm{O}\left(\frac{\mathrm{e}^t}{i^2}\right), \qquad \int_a^b \tilde{c}(s)|I(s)|^2\,\mathrm{d}s < c^2,$$

$$||\delta_{ij} + (\mathrm{e}^{t\xi_i} - 1) \int_0^x g_i(w,q(w),\lambda_i) g_j(w,q(w),\lambda_i)\,\mathrm{d}w||_1 < 1,$$

where $c := |\int_a^b I(s)\,\mathrm{d}s|$, *the function* $\tilde{c}(s)$ *is given in the proof and* $x \in]0,1[$, *where* x *is the parameter in the normalized eigenfunctions* $g_k(x,)$. *If* $x = 0$ *or* $x = 1$, *then there exists no* t *such that the above conditions for the fixed point equations hold.*

Proof. To apply the Banach fixed point theorem we have to show that G is a contraction. We have

$$\begin{aligned} &||G(q_1) - G(q_2)||_2^2 \\ &= \frac{1}{c^2} || \int_a^b I(s)\Xi(x,s,q_1)\mathrm{d}s - \int_a^b I(s)\Xi(x,s,q_2)\,\mathrm{d}s||_2^2 \\ &\leq \frac{1}{c^2} \int_a^b |I(s)|^2 \int_0^1 |\Xi(x,s\xi,q_1)\,\mathrm{d}s - \Xi(x,s\xi,q_2)|^2\,\mathrm{d}x\,\mathrm{d}s, \end{aligned} \tag{4.13}$$

where

$$c := |\int_a^b I(s)\,\mathrm{d}s|.$$

Let $H(v) := H(x, vq_1 + (1-v)q_2)$, $v \in [0, 1]$ be the operator which interpolates between $\Theta(x, s\xi, q_1)$ and $\Theta(x, s\xi, q_2)$, i.e.

$$H(x, vq)|_{v=1} = \Theta(x, s\xi, q_1), \qquad H(x, vq)|_{v=0} = \Theta(x, s\xi, q_2). \tag{4.14}$$

If $\Xi(x, s\xi, q_1)\Xi^{-1}(x, s\xi, q_2)$ is trace class, then

$$\det \Theta(x, s\xi, q_1)\Theta^{-1}(x, s\xi, q_2) = -\exp \operatorname{tr} \int_0^1 \frac{\mathrm{d}H(v)}{\mathrm{d}v} H^{-1}(v)\,\mathrm{d}v. \tag{4.15}$$

Since $\Theta(x, s\xi, q_1) = 1 + \text{trace class}$ and

$$\Theta^{-1}(x, s\xi, q_2) = 1 + \text{trace class} = \sum_{k=0}^{\infty}(-1)^k B^k,$$

if $||B||_1 < 1$, a sufficient condition for $\Theta(x, s\xi, q_1)\Theta^{-1}(x, s\xi, q_2)$ to be trace class is that the matrix $B_{ij} = (\mathrm{e}^{t\xi_i} - 1)\int_0^x g_i(w, q(w), \lambda_i)g_j(w, q(w), \lambda_i)\,\mathrm{d}w$ has trace class norm less than one. Therefore,

$$\begin{aligned}
&||\Xi(x, s\xi, q_1)\,\mathrm{d}s - \Xi(x, s\xi, q_2)||_2^2 \\
&\quad = ||2\frac{\mathrm{d}^2}{\mathrm{d}x^2}\log\det\Theta(x, s\xi, q_1)\Theta^{-1}(x, s\xi, q_2)||_2^2 = ||2\frac{\mathrm{d}^2}{\mathrm{d}x^2}\log\det\Theta_1\Theta_2^{-1}||_2^2 \\
&\quad \le 2\int_0^1 |\frac{\mathrm{d}^2}{\mathrm{d}x^2}\log\det\Theta_1\Theta_2^{-1}|^2\,\mathrm{d}x \le 2\int_0^1 |\frac{\mathrm{d}^2}{\mathrm{d}x^2}\operatorname{tr}\int_0^1 \frac{\mathrm{d}H(v)}{\mathrm{d}v}H^{-1}(v)\,\mathrm{d}v|^2\,\mathrm{d}x.
\end{aligned} \tag{4.16}$$

Using

$$\begin{aligned}
\frac{\mathrm{d}H(v)}{\mathrm{d}v} &= (\mathrm{e}^{t\xi_i} - 1)\int_0^x \frac{\mathrm{d}}{\mathrm{d}v}[g_i(w, vq_1(w) + (v-1)q_2(w)) \\
&\qquad \times g_j(w, vq_1(w) + (v-1)q_2(w))](q_1(w) - q_2(w))\,\mathrm{d}w \\
&=: (\mathrm{e}^{t\xi_i} - 1)\int_0^x \frac{\mathrm{d}}{\mathrm{d}v}[g_ig_j](q_1(w) - q_2(w))\,\mathrm{d}w, \\
H(v)_{ij}^{-1} &= \sum_{k=0}^{\infty}(1 - \mathrm{e}^{t\xi_i})^k \left(\int_0^x [g_i(w, vq_1(w) + (v-1)q_2(w)) \right. \\
&\qquad \left. \times g_j(w, vq_1(w) + (v-1)q_2(w))]\,\mathrm{d}w\right)^k \\
&=: \sum_{k=0}^{\infty}(1 - \mathrm{e}^{t\xi_i})^k \left(\int_0^x g_ig_j\,\mathrm{d}w\right)^k,
\end{aligned}$$

we get

$$
\begin{aligned}
&2\int_0^1 |\frac{\mathrm{d}^2}{\mathrm{d}x^2}\,\mathrm{tr}\int_0^1 \frac{\mathrm{d}H(v)}{\mathrm{d}v}H^{-1}(v)\,\mathrm{d}v|^2\,\mathrm{d}x \\
&= 2\int_0^1 |\frac{\mathrm{d}^2}{\mathrm{d}x^2}\,\mathrm{tr}\int_0^1 (\mathrm{e}^{t\xi_i}-1)\int_0^x \frac{\mathrm{d}}{\mathrm{d}v} g_i(v,x)g_j(v,x)(q_1(x)-q_2(x))\,\mathrm{d}w \\
&\quad \times \sum_{k=0}^{\infty}(1-\mathrm{e}^{t\xi_i})^k\left(\int_0^x g_ig_j\,\mathrm{d}w\right)^k \mathrm{d}v|^2\,\mathrm{d}x \\
&\le 2\int_0^1\left\{\sum_{ij}\int_0^1\left((\mathrm{e}^{t\xi_i}-1)\frac{\mathrm{d}^2}{\mathrm{d}v\,\mathrm{d}x}[g_ig_j](q_1(x)-q_2(x))\right.\right. \\
&\quad \times \sum_{k=0}^{\infty}(1-\mathrm{e}^{t\xi_j})^{2k}\left(\int_0^x g_ig_j\,\mathrm{d}w\right)^{2k} \\
&\quad +(\mathrm{e}^{t\xi_i}-1)\int_0^x [g_ig_j](q_1(w)-q_2(w))\,\mathrm{d}w \\
&\quad \times \sum_{k=0}^{\infty}(1-\mathrm{e}^{t\xi_j})^{2k}2k\left(\int_0^x g_ig_j\,\mathrm{d}w\right)^{2k-2}\left[(2k-1)(g_i(x)g_j(x))^2\right. \\
&\quad \left.+\int_0^x (g_ig_j)\,\mathrm{d}w\frac{\mathrm{d}}{\mathrm{d}x}(g_ig_j)\right] + 2(\mathrm{e}^{t\xi_i}-1)\frac{\mathrm{d}}{\mathrm{d}v}[g_ig_j](q_1(w)-q_2(w))\,\mathrm{d}w \\
&\quad \left.\left.\times\sum_{k=0}^{\infty}(1-\mathrm{e}^{t\xi_j})^{2k}2k\left(\int_0^x g_ig_j\,\mathrm{d}w\right)^{2k-1} g_i(x)g_j(x)\right)\mathrm{d}x\,\mathrm{d}v\right\} \\
&=: A+B+C.
\end{aligned}
$$

We bound A by using the Cauchy–Schwarz inequality and the Hölder inequality,

$$
\begin{aligned}
A = 2\int_0^1\sum_{ij}\int_0^1 &\left((\mathrm{e}^{t\xi_i}-1)\frac{\mathrm{d}^2}{\mathrm{d}v\,\mathrm{d}x}[g_ig_j](q_1(x)-q_2(x))\right. \\
&\left.\times\sum_{k=0}^{\infty}(1-\mathrm{e}^{t\xi_j})^{2k}\left(\int_0^x g_ig_j\,\mathrm{d}w\right)^{2k}\right)\mathrm{d}v\,\mathrm{d}x
\end{aligned}
$$

$$\leq \sum_{ij} 2 \int_0^1 |(e^{t\xi_i} - 1)|^2 \sum_{k=0}^{\infty} |(1 - e^{t\xi_j})|^{2k}$$

$$\times \left(\int_0^1 |\frac{d^2}{dv\, dx}[g_i g_j](q_1(x) - q_2(x))|^2 dv \right) dx \left(\int_0^1 |g_i|^2 \right)^k \left(\int_0^1 |g_j|^2 \right)^k .$$

Since,

$$\frac{d}{ds} g_i = \frac{d}{ds} g_i(x, sq_1 + (1-s)q_2) = O\left(\frac{1}{i}\right), \quad g_i = x\sqrt{2}\sin \pi i x + O(1/i),$$

we get for the A-term,

$$A \leq \sum_{ij} 2 \int_0^1 |(e^{t\xi_i} - 1)|^2 \sum_{k=0}^{\infty} |(1 - e^{t\xi_j})|^{2k}$$

$$\times \left(O\left(\frac{1}{i}\right) + O\left(\frac{1}{j}\right)\left[||q_1 - q_2||_2^2 \right] + \int_0^1 \frac{d}{dx}(q_1 - q_2)\, dv \right)$$

$$\times \left(\int_0^1 |g_i|^2 \right)^k \left(\int_0^1 |g_j|^2 \right)^k$$

$$\leq \sum_{ij} 2 \int_0^1 |(e^{t\xi_i} - 1)|^2 \frac{1}{1 - |(1 - e^{t\xi_j})| \int_0^1 |g_i|^4 \int_0^1 |g_j|^4}$$

$$\times \left(||q_1 - q_2||_2^2 \right) (O(i) + O(j))$$

if $|(1 - e^{t\xi_j})| \int_0^1 |g_i|^4 \int_0^1 |g_j|^4 < 1$. If we bound

$$\frac{O(j)}{1 - |(1 - e^{t\xi_j})| \int_0^1 |g_i|^4 \int_0^1 |g_j|^4} \leq O\left(\frac{e^t}{j^2}\right),$$

then we can perform the j-sum, i.e.

$$A \leq \sum_i 2 \int_0^1 |(e^{t\xi_i} - 1)|^2 O(e^t) O(i)[||q_1 - q_2||_2^2] \leq c_2(t)||q_1 - q_2||_2^2,$$

where we assumed that $|e^{t\xi_i} - 1|O(i) < O(e^t/i^2)$. The other terms B and C are treated in the same way and we get with the above made assumptions that

$$||G(q_1) - G(q_2)||_2^2 \leq \frac{1}{c^2} \int_a^b \tilde{c}(s)|I(s)|^2\, ds ||q_1 - q_2||_2^2 =: c_3||q_1 - q_2||_2^2, \qquad (4.17)$$

where $\bar{c}(s)$ is the maximum of the three constants coming from the A, B and C term, respectively, and in order that we have a contraction c_3 has to smaller than one, i.e.

$$\int_a^b \bar{c}(s)|I(s)|^2\,\mathrm{d}s < c^2. \tag{4.18}$$

The exclusion of the parameter values $x = 0, 1$ is discussed after the following corollary in an example. Hence, up to this last point, Theorem 6 is proved. □

Corollary 1. *If a curve $c^t : [a, b] \to M(e)$ satisfies all the assumptions of Theorem* 6, *that is the interval $[a, b]$, the points $q \in M(e)$ and the vector fields V_ξ all satisfy the conditions of Theorem* 6, *then it is a geodesic.*

Example. Take $q = 0$. Then, $g_j = \sin j\pi x$. The condition $|(1-\mathrm{e}^{t\xi_j})|\int_0^1 |g_i|^4 \int_0^1 |g_j|^4 < 1$ in Theorem 6 then reads

$$t < \frac{1}{|\xi_j|}\left(\left\{\frac{3}{8}(x-1) + \frac{1}{16\pi^2 i^2 x}\left[\frac{\sin i\pi 4x}{2} - \sin i\pi 2x\right]\right\}^{-1}\right.$$
$$\left.\times\left\{\frac{3}{8}(x-1) + \frac{1}{16\pi^2 j^2 x}\left[\frac{\sin j\pi 4x}{2} - \sin j\pi 2x\right]\right\}^{-1} - 1\right),$$

which has no solution if $x = 0, 1$ such that $t > 0$.

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